EXPANDERS, RANK AND GRAPHS OF GROUPS

MARC LACKENBY

1. Introduction

A central principle of this paper is that, for a finitely presented group G, the algebraic properties of its finite index subgroups should be reflected by the geometry of its finite quotients. These quotients can indeed be viewed as geometric objects, in the following way. If we pick a finite set of generators for G, these map to a generating set for any finite quotient and hence endow this quotient with a word metric. This metric of course depends on the choice of generators, but if we were to pick another set of generators for G, the metrics on the quotients would change by a bounded factor. Thus, although the metric on any given finite quotient is unlikely to be useful, the metrics on the whole collection of finite quotients have a good deal of significance.

The above principle is inspired by a common theme in manifold theory: that the geometry and topology of a Riemannian manifold should relate to its fundamental group. Thus, if we pick a compact Riemannian manifold with fundamental group G (which is possible since G is finitely presented), and let M_i be the covering space corresponding to a finite index normal subgroup G_i, then the geometry of M_i should have consequences for the algebraic structure of G_i. But M_i is coarsely approximated by the word metric on the quotient G/G_i.

In this paper, we will focus on the geometric properties of the quotient groups that relate to their Cheeger constant. Recall that this is defined as follows. Fix a finite set S of generators for G, and let X_i be the Cayley graph of G/G_i with respect to S. The *Cheeger constant* of X_i, denoted $h(X_i)$, is defined to be

$$\min\left\{\frac{|\partial A|}{|A|} : A \subset V(X_i) \text{ and } 0 < |A| \leq |V(X_i)|/2\right\}.$$

Here, $V(X_i)$ is the vertex set of X_i, and ∂A denotes the set of edges joining a vertex in A to one not in A. The group G is said to have *Property* (τ) with respect to a collection $\{G_i\}$ of finite index normal subgroups if the Cheeger constants $h(X_i)$ are bounded away from zero. The graphs X_i then form what is known as an *expanding family* or an *expander*. There is a remarkably rich theory relating to Property (τ) [7]. Possibly its most striking aspect is that it has so many equivalent definitions, drawing on many different areas of mathematics, including graph theory, differential geometry and representation theory. It is, in general, very difficult to construct explicit expanding

families of graphs with bounded valence ([7],[10]). A consequence of this paper is that they are probably very common.

We will focus mainly on two algebraic properties of the subgroups G_i. The first is whether or not G_i decomposes as an amalgamated free product or HNN extension; in other words, whether or not G_i admits a non-trivial decomposition as a graph of groups. Such groups play a central rôle in combinatorial group theory. The second is a new concept related to their rank. The *rank* of a group G is the minimal size of a generating set, and is denoted $d(G)$. When G_i is a finite index subgroup of G, the Reidermeister-Schreier process [9] gives of a collection of $(d(G)-1)[G : G_i]+1$ generators for G_i. Thus,

$$d(G_i) - 1 \leq (d(G) - 1)[G : G_i].$$

When $\{G_i\}$ is a collection of finite index subgroups of G, the *rank gradient* of the pair $(G, \{G_i\})$ measures the strictness of this inequality. It is defined to be

$$\inf_i \left\{ \frac{d(G_i) - 1}{[G : G_i]} \right\}.$$

The *rank gradient* of G is defined by taking $\{G_i\}$ to be the set of all its finite index normal subgroups.

A collection $\{G_i\}$ of subgroups of a group G is termed a *lattice* if, whenever G_i and G_j are in the collection, so is $G_i \cap G_j$. We can now state the main theorem of this paper.

Theorem 1.1. *Let G be a finitely presented group, and let $\{G_i\}$ be a lattice of finite index normal subgroups. Then at least one of the following holds:*

1. *G_i is an amalgamated free product or HNN extension for infinitely many i;*
2. *G has Property (τ) with respect to $\{G_i\}$;*
3. *the rank gradient of $(G, \{G_i\})$ is zero.*

We will prove this theorem in §2.

It is conclusion (3) in Theorem 1.1 that is least familiar. It raises the question of which groups have non-zero rank gradient and which do not. In §3, we will investigate rank gradient quite thoroughly. It seems likely that, among abstract groups in general, those with zero rank gradient are rather special, because they have a sequence of finite index subgroups with relatively small generating sets not arising from the Reidermeister-Schreier process. However, there are large classes of groups with zero rank gradient. For example, mapping tori always have zero rank gradient. Another class consists of the S-arithmetic groups for which the congruence kernel is trivial. A familiar example is

$\mathrm{SL}(n,\mathbb{Z})$, for $n > 2$. A third class arises from the following result, which we shall prove in §3. This is a consequence of a slightly strengthened version of Theorem 1.1.

Theorem 1.2. *Any finitely presented, residually finite, amenable group has non-positive rank gradient.*

On the other hand, to prove that a given group has non-zero rank gradient is usually rather hard, since it is often difficult to find good lower bounds on the rank of a group. An obvious class of examples are groups with deficiency more than one. We shall also show that the free product of two non-trivial groups (not both $\mathbb{Z}/2\mathbb{Z}$) has non-zero rank gradient.

It should not be assumed that the three possible conclusions in Theorem 1.1 are mutually exclusive. In §4, we investigate the possible combinations of these properties that can arise.

Theorem 1.1 is a group-theoretic version of a result in 3-manifold theory: Theorem 1.7 of [4]. Instead of dealing with rank gradient, this used a related notion, the Heegaard gradient of a 3-manifold, which measures the rate at which the Heegaard genus of the manifold's finite-sheeted covering spaces grows as a function of the covering degree. The purpose of Theorem 1.7 in [4] is that it represents part of a programme for proving the virtually Haken conjecture, which is a key unsolved problem about 3-manifolds. This asserts that, when G is the fundamental group of a closed hyperbolic 3-manifold, some finite index subgroup G_i should admit a non-trivial decomposition as a graph of groups.

This is the first in a pair of papers, which use the geometry and topology of finite Cayley graphs as a tool in group theory. In the second [5], we prove the following purely algebraic result.

Theorem 1.3. *Let G be a finitely presented group. Then the following are equivalent:*

1. *some finite index subgroup of G admits a surjective homomorphism onto a non-abelian free group;*
2. *there exists a sequence $G_1 \geq G_2 \geq \dots$ of finite index subgroups of G, each normal in G_1, such that*
   - *(i)* G_i/G_{i+1} *is abelian for all* $i \geq 1$*;*
   - *(ii)* $\lim_{i\to\infty}((\log[G_i : G_{i+1}])/[G : G_i]) = \infty$*;*
   - *(iii)* $\limsup_i(d(G_i/G_{i+1})/[G : G_i]) > 0$.

The difficult part of this theorem is the implication (2) ⇒ (1). In §5, we establish

a partial result in this direction. Using Theorem 1.1, we show that (2) implies that G_i is a non-trivial graph of groups for all sufficiently large i. The proof in [5] of Theorem 1.3 uses an extension of the ideas behind Theorem 1.1.

2. Proof of the main theorem

The following appears as Lemmas 2.1 and 2.2 in [4].

Lemma 2.1. *Let X be a Cayley graph of a finite group, and let D be a non-empty subset of $V(X)$ such that $|\partial D|/|D| = h(X)$ and $|D| \leq |V(X)|/2$. Then $|D| > |V(X)|/4$. Furthermore, the subgraphs induced by D and its complement D^c are connected.*

Proof of Theorem 1.1. Suppose that (2) and (3) of 1.1 do not hold. Our aim is to show that (1) must be true.

We fix ϵ to be some real number strictly between 0 and $\frac{2}{\sqrt{3}} - 1$, but where we view it as very small. Since the rank gradient of $(G, \{G_i\})$ is non-zero, there is a subgroup H in the lattice $\{G_i\}$ such that $(d(H) - 1)/[G : H]$ is at most $(1 + \epsilon)$ times the rank gradient of $(G, \{G_i\})$. The pair $(H, \{G_i \cap H\})$ has rank gradient at least $[G : H]$ times the rank gradient of $(G, \{G_i\})$, since $\{G_i \cap H\}$ is a sublattice of $\{G_i\}$. So, $d(H) - 1$ is at most $(1 + \epsilon)$ times the rank gradient of $(H, \{G_i \cap H\})$. Also, (2) does not hold for this sublattice. Hence, by replacing G by H, and replacing $\{G_i\}$ by $\{G_i \cap H\}$, we may assume that $d(G) - 1$ is at most $(1 + \epsilon)$ times the rank gradient of $(G, \{G_i\})$.

Let K be a finite 2-complex having fundamental group G, arising from a minimal generator presentation of G. Thus, K has a single vertex and $d(G)$ edges. Let L be the sum of the lengths of the relations in this presentation. Let $K_i \to K$ be the covering corresponding to G_i, and let X_i be the 1-skeleton of K_i. Since we are assuming that G does not have Property (τ) with respect to $\{G_i\}$, we may pass to a sublattice where $h(X_i) \to 0$. For each i, let D_i be a non-empty subset of $V(X_i)$ such that $|\partial D_i|/|D_i| = h(X_i)$ and $|D_i| \leq |V(X_i)|/2$. Lemma 2.1 asserts that $|D_i| > |V(X_i)|/4$. We will use D_i to construct a decomposition of K_i into two overlapping subsets. Let A_i (respectively, B_i) be the closure of the union of those cells in K_i that intersect D_i (respectively, D_i^c). Let C_i be $A_i \cap B_i$. Lemma 2.1 asserts that the subgraphs induced by D_i and D_i^c are connected, and hence so are A_i and B_i. The 1-skeleton of A_i consists of three types of edges (that are not mutually exclusive):

(i) those edges with both endpoints in D_i,

(ii) the edges in ∂D_i,

(iii) those edges in the boundary of a 2-cell that intersects both D_i and D_i^c.

If we consider the $d(G)$ oriented edges of K_i emanating from the identity vertex in K_i and translate these edges by the covering translations in D_i (where we view D_i as subset of G/G_i), we will cover every edge in (i), and possibly others. Hence, there are at most $|D_i|d(G)$ edges of type (i). Similarly, any type (iii) edge lies in a 2-cell that intersects both D_i and D_i^c. Place the basepoint of this 2-cell at one its vertices in D_i that is adjacent to ∂D_i. Translating this basepoint to the identity vertex, we obtain a corner of a 2-cell incident to the identity vertex. There are at most L such corners. The 2-cell runs over at most L 1-cells, and there are at most $|\partial D_i|$ possibilities for the translation. So, there are no more than $|\partial D_i|L^2$ type (iii) edges. There are $|\partial D_i|$ type (ii) edges, and so, there are at most $|\partial D_i|(L^2+1)$ type (ii) and (iii) edges in total. Since $d(\pi_1 A_i) - 1$ is at most the number of edges of A_i minus the number of its vertices,

$$\begin{aligned} d(\pi_1 A_i) - 1 &\leq |D_i|d(G) + |\partial D_i|(L^2+1) - |D_i| \\ &= |D_i|(d(G) - 1 + h(X_i)(L^2+1)) \\ &\leq \tfrac{1}{2}[G : G_i](d(G) - 1 + h(X_i)(L^2+1)) \\ &\leq \tfrac{1}{2}(1+\epsilon)[G : G_i](d(G) - 1) \text{ when } h(X_i) \text{ is sufficiently small} \\ &\leq \tfrac{1}{2}(1+\epsilon)^2(d(G_i) - 1). \end{aligned}$$

A similar inequality holds for $d(\pi_1 B_i) - 1$, but where $\frac{1}{2}$ is replaced throughout by $\frac{3}{4}$. We also note that the 1-skeleton of C_i consists of type (ii) and type (iii) edges (that may also be of type (i)), and so, for each component $C_{i,j}$, of C_i,

$$d(\pi_1 C_{i,j}) \leq |\partial D_i|(L^2+1) = |D_i|h(X_i)(L^2+1) \leq \tfrac{1}{2}[G : G_i]h(X_i)(L^2+1).$$

If C_i is disconnected, then G_i is an HNN extension, giving (1). Thus, we may assume that C_i is connected. Then, by the Seifert-Van Kampen theorem, $\pi_1 K_i$ $(= G_i)$ is the pushout of the diagram

$$\begin{array}{ccc} \pi_1 C_i & \longrightarrow & \pi_1 A_i \\ \big\downarrow & & \\ \pi_1 B_i & & \end{array}$$

where the maps are induced by inclusion. These homomorphisms need not be injective. However, if we write $\mathrm{Im}(\pi_1 C_i)$ for the image of $\pi_1 C_i$ in $\pi_1 K_i$, and so on, then $\pi_1 K_i$ is the pushout of

$$\begin{array}{ccc} \mathrm{Im}(\pi_1 C_i) & \longrightarrow & \mathrm{Im}(\pi_1 A_i) \\ \big\downarrow & & \\ \mathrm{Im}(\pi_1 B_i). & & \end{array}$$

This follows from a straightforward application of the universal property of pushouts. The maps in the above diagram are now injections. When $h(X_i)$ is sufficiently small, neither $\mathrm{Im}(\pi_1 A_i)$ nor $\mathrm{Im}(\pi_1 B_i)$ can be all of G_i. This is because they then have rank at most $\frac{3}{4}(1+\epsilon)^2(d(G_i)-1)+1$, which is less than $d(G_i)$, when i is sufficiently large, by our assumption that $\epsilon < \frac{2}{\sqrt{3}} - 1$. Thus, we deduce that G_i is the non-trivial amalgamated free product of $\mathrm{Im}(\pi_1 A_i)$ and $\mathrm{Im}(\pi_1 B_i)$ along $\mathrm{Im}(\pi_1 C_i)$. $\square$

The argument above gives rather more, in fact, than is stated in Theorem 1.1. It immediately implies the following.

Addendum 2.2. *Theorem 1.1 remains true if (1) is replaced by:*

1′. *G_i is an amalgamated free product $P_i *_{R_i} Q_i$ or HNN extension $P_i *_{R_i}$ for infinitely many i, and in some subsequence,*

$$\lim_{i\to\infty} \frac{d(R_i)}{d(G_i)} = 0.$$

Furthermore, in the case when these G_i are amalgamated free products $P_i *_{R_i} Q_i$, we may ensure that the following also hold in this subsequence:

$$\frac{1}{4} \leq \liminf_i \frac{d(P_i)}{d(G_i)} \leq \limsup_i \frac{d(P_i)}{d(G_i)} \leq \frac{3}{4}$$
$$\frac{1}{4} \leq \liminf_i \frac{d(Q_i)}{d(G_i)} \leq \limsup_i \frac{d(Q_i)}{d(G_i)} \leq \frac{3}{4}.$$

3. Rank gradient

The first examples one comes to of groups with non-zero rank gradient are free non-abelian groups. If G_i is a finite index subgroup of a finitely generated free group F, then

$$d(G_i) - 1 = (d(F) - 1)[F : G_i],$$

and so the rank gradient of F is $d(F) - 1$.

Since $\mathrm{SL}(2, \mathbb{Z})$ has a free non-abelian normal subgroup of finite index, the following lemma implies that it has non-zero rank gradient.

Lemma 3.1. *Let H be a finite index normal subgroup of a finitely generated infinite group G, and let $\{G_i\}$ be a collection of finite index normal subgroups of G. Then $(G, \{G_i\})$ has non-zero rank gradient if and only if $(H, \{G_i \cap H\})$ has non-zero rank gradient. Hence, G has non-zero rank gradient if and only if the same is true of H.*

Proof. This is a consequence of the following inequalities:

$$\frac{[G:G_i]}{[G:H]} \leq [H:G_i \cap H] \leq [G:G_i]$$
$$d(G_i) - d(G_i/G_i \cap H) \leq d(G_i \cap H) \leq (d(G_i)-1)[G_i:G_i \cap H]+1.$$

We note that $G_i/G_i \cap H$ is a finite group with order at most $[G:H]$, and hence $d(G_i/G_i \cap H)$ is bounded, independent of i. □

The same lemma gives the following more general conclusion. Let G be the amalgamated free product $A *_C B$, where A and B are finite, where C is a proper subgroup of both A and B, and where at least one of $[A:C]$ and $[B:C]$ is more than two. Then G has non-zero rank gradient.

There are two possible generalisations from free non-abelian groups. The first is to groups with deficiency more than one, namely those groups G admitting a finite presentation $\langle X|R\rangle$ where $|X| > |R|+1$. If we apply the Reidermeister-Schreier process to a finite index subgroup G_i, we obtain a presentation for G_i with $(|X|-1)[G:G_i]+1$ generators and $|R|[G:G_i]$ relations. Hence, the first Betti number of G_i is at least

$$(|X|-1-|R|)[G:G_i]+1.$$

This is a lower bound for its rank, and so the rank gradient of G is at least $(|X|-1-|R|)$, which is positive.

The second way to generalise from the example of free non-abelian groups is to free products of groups. Here, we have the following result.

Proposition 3.2. *Let G be the free product of two non-trivial, finitely generated groups, not both isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Then G has non-zero rank gradient.*

Proof. Let $G = A * B$, where $|A| > 2$ and $|B| > 1$. This gives a graph of groups decomposition of G. This lifts to a graph of groups decomposition for any finite index normal subgroup G_i. The vertex groups covering A are all isomorphic to $A \cap G_i$. There are $[G:G_i]/[A:A \cap G_i]$ such vertices. A similar statement holds for the vertices covering B. Since the edge group of G is trivial, so too are all the edge groups of G_i, and there are $[G:G_i]$ of them. Hence, the first Betti number of the graph for G_i is

$$[G:G_i]\left(1-\frac{1}{[A:A\cap G_i]}-\frac{1}{[B:B\cap G_i]}\right)+1.$$

This is a lower bound for the rank of G_i. Thus, the rank gradient of G is positive, unless one of $[A:A\cap G_i]$ and $[B:B\cap G_i]$ is one for infinitely many i *or* they are both two for

infinitely many i. In the former case, the vertex groups of G_i that cover the A (or B) group are themselves isomorphic to A (or B), and there are $[G : G_i]$ of them. In this case, G_i is a free product, with at least $[G : G_i]$ summands isomorphic to A (or B). By Grushko's theorem [9], the rank of G_i is then at least $[G : G_i]d(A)$ (or $[G : G_i]d(B)$), and so G has non-zero rank gradient. Now consider the second case, where $[A : A \cap G_i]$ and $[B : B \cap G_i]$ are both two. Since A has more than two elements, $A \cap G_i$ is non-trivial. The vertex groups of G_i covering A are all isomorphic to $A \cap G_i$, and there are $[G : G_i]/2$ of them. Hence, $d(G_i) \geq d(A \cap G_i)[G : G_i]/2$, and again G has non-zero rank gradient. $\square$

The condition in the above theorem that G not be isomorphic to $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ is clearly necessary. This is because $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ contains $\mathbb{Z}$ as a finite index normal subgroup, and hence has zero rank gradient, by Lemma 3.1

There is another source of examples of groups having collections of subgroups with non-zero rank gradient.

Proposition 3.3. *Let G be a finitely generated group that admits a surjective homomorphism $\phi: G \to F$ onto a free non-abelian group. Then, for any collection $\{G_i\}$ of finite index subgroups that each contain the kernel of ϕ, $(G, \{G_i\})$ has non-zero rank gradient.*

Proof. For any such G_i, $[G : G_i] = [F : \phi(G_i)]$. Hence,

$$d(G_i) \geq d(\phi(G_i)) = (d(F) - 1)[F : \phi(G_i)] + 1 = (d(F) - 1)[G : G_i] + 1.$$

So, the rank gradient of $(G, \{G_i\})$ is at least $d(F) - 1$. $\square$

This is relevant in 3-manifold theory.

Corollary 3.4. *Let M be a compact irreducible 3-manifold with non-empty boundary, other than an I-bundle over a disc, annulus, torus or Klein bottle. Then $\pi_1 M$ has an infinite lattice of finite index subgroups with non-zero rank gradient. Hence, the corresponding covers have non-zero Heegaard gradient.*

Proof. $\pi_1 M$ has a finite index normal subgroup that admits a surjective homomorphism ϕ onto $\mathbb{Z} * \mathbb{Z}$, by a theorem of Cooper, Long and Reid [1]. The finite index subgroups of $\pi_1 M$ that contain the kernel of ϕ form the required infinite lattice. The final statement of the corollary follows from the observation that the Heegaard genus of a 3-manifold is at least the rank of its fundamental group. $\square$

We now turn to groups with zero rank gradient. The first collection of examples

are mapping tori. These are constructed from a finitely generated group A and a homomorphism $\phi\colon A \to A$. The associated *mapping torus* G is

$$\langle A, t \mid a = t^{-1}\phi(a)t \text{ for all } a \in A\rangle.$$

Its rank is at most $d(A) + 1$. It admits a surjective homomorphism $G \to \mathbb{Z}$, sending A to 0, and t to 1. Compose this with the homomorphism to $\mathbb{Z}/n\mathbb{Z}$, reducing the integers modulo n. The kernel of this homomorphism is a group G_n, which is an index n normal subgroup of G. It is isomorphic to the mapping torus associated with A and ϕ^n, and hence has rank at most $d(A) + 1$. Thus, the collection $\{G_n\}$ has bounded rank, and hence $(G, \{G_n\})$ has zero rank gradient.

Our second class of groups with zero rank gradient is the S-arithmetic groups with trivial congruence kernel. It is a result of Sury and Venkataramana [13] that, for such a group G, there is uniform bound on the rank of its principal congruence subgroups. In [13], they proved this in the case of $\mathrm{SL}(n, \mathbb{Z})$, where $n \geq 3$, but they state that the proof carries over to this much larger class of groups. Hence, G has zero rank gradient. I am grateful to the referee for suggesting that we consider lattices other than $\mathrm{SL}(n, \mathbb{Z})$ (with $n > 2$). An interesting further collection of examples comes from the following result.

Theorem 1.2. *Any finitely presented, residually finite, amenable group has non-positive rank gradient.*

Proof. Let G be such a group, which we may assume is infinite. Let $\{G_i\}$ be its finite index normal subgroups. Since G is infinite, amenable and residually finite, it does not have Property (τ), and so (2) of Theorem 1.1 cannot hold. Suppose that $(1')$ of Addendum 2.2 holds. Then, for infinitely many i, G_i is a graph of groups and hence acts cocompactly on a tree. Each vertex of this tree has stabiliser which is a conjugate of P_i or Q_i. The number of edges emanating from this vertex is the index of R_i in P_i or Q_i, as appropriate. By $(1')$ this index can be arbitrarily large, and so the tree is not homeomorphic to the real line. Hence, G contains a non-abelian free group, which contradicts the assumption that it is amenable. We deduce that (3) holds: G has zero rank gradient. □

We close this section with examples of groups, each having an infinite lattice of subgroups with non-zero rank gradient, and another infinite lattice of subgroups with zero rank gradient. Such groups arise as the fundamental group of a hyperbolic 3-manifold that fibres over the circle, with a fibre a compact surface with negative Euler characteristic and non-empty boundary. Since these groups are mapping tori, they have zero rank gradient. But Corollary 3.4 also gives that they have non-zero rank gradient

with respect to some infinite lattice of finite index subgroups. A concrete example is the fundamental group of the figure-eight knot complement [14]. This is an index 24 subgroup of $\mathrm{SL}(2, \mathcal{O}_3)$, where $\mathcal{O}_3$ is the ring of integers in $\mathbb{Q}(\sqrt{-3})$.

4. Examples

In this section, we investigate which possible combinations of (1), (2) and (3) of Theorem 1.1 can arise.

Example 4.1. $\mathbb{Z}$ is trivially an HNN extension. It does not have Property (τ). And it has zero rank gradient. Thus it satisfies (1) and (3) but not (2). More generally, any mapping torus has these properties.

Example 4.2. A non-abelian free group has non-zero rank gradient. It admits a surjective homomorphism onto $\mathbb{Z}$ and hence does not have Property (τ). Any finite index subgroup is free, and is, in particular, a non-trivial graph of groups. So, it satisfies (1), but not (2) or (3).

Example 4.3. $\mathrm{SL}(2, \mathbb{Z})$ is an amalgamated free product $\mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$, and hence any finite index subgroup is a non-trivial graph of groups. It has Property (τ) with respect to its congruence subgroups [7]. And we have already seen that it has non-zero rank gradient. Thus, the lattice of congruence subgroups satisfies (1) and (2) but not (3).

Example 4.4. $\mathrm{SL}(n, \mathbb{Z})$, where $n \geq 3$, has Property (T). Hence, no finite index subgroup is either an HNN extension or an amalgamated free product. Another consequence is that it has Property (τ). We have already seen that it has zero rank gradient. So, these groups satisfy (2) and (3) but not (1).

Example 4.5. $\mathrm{SL}(2, \mathbb{Z}[\frac{1}{p}])$, where p is a prime, satisfies (1), (2) and (3). It was proved by Serre [12] that $\mathrm{SL}(2, \mathbb{Z}[\frac{1}{p}])$ decomposes as a non-trivial graph of groups, and hence, so does any finite index subgroup. It has infinitely many finite index subgroups, for example, its principal congruence subgroups, and hence it satisfies (1). In fact, any finite index subgroup is contained in a principal congruence subgroup [11]. In other words, it has trivial congruence kernel, and so, by the remarks about S-arithmetic groups in §3, it satisfies (3). Also using this control on its finite index subgroups, an argument of Lubotzky in [7] gives that it has Property (τ), establishing (2). I am grateful to the referee for suggesting this example.

Example 4.6. The Grigorchuk group Γ is finitely generated, infinite, torsion, amenable, and residually finite [3]. It therefore satisfies neither (1) nor (2). We claim also that Γ has zero rank gradient, and so that (3) is true. For each positive integer k, Γ has a 'level k congruence subgroup' $St_\Gamma(k)$, which is normal and has finite index [2]. For any $k \geq 4$, $St_\Gamma(k)$ is isomorphic to the product of 2^{k-3} copies of $St_\Gamma(3)$. (This follows from Proposition 30 (iii) and (vi) in [2]). So, $d(St_\Gamma(k)) \leq 2^{k-3} d(St_\Gamma(3))$. However, the index of $St_\Gamma(k)$ in Γ is $2^{5 \cdot 2^{k-3}+2}$. Thus, these subgroups have zero rank gradient.

However, Γ is not a finitely presented example. There are no known examples of a finitely presented group that fails to have Property (τ) but that does not have a finite index subgroup with infinite abelianisation. As a result, there are no known finitely presented groups where neither (1) nor (2) hold. Whether such groups can exist is an important unresolved problem.

There is one other theoretical possibility: where (2) holds but (1) and (3) do not. The difficulty here is the problem of establishing that a group has non-zero rank gradient if it is not an amalgamated free product. It seems likely that the absence of known examples here is merely due to a lack of mathematical tools, rather than due to genuine non-existence.

5. Finite index subgroups having free non-abelian quotients

In this section, we prove the following result, which is a weaker form of Theorem 1.3.

Theorem 5.1. *Let G be a finitely presented group. Suppose that there exists a sequence $G_1 \geq G_2 \geq \dots$ of finite index subgroups of G, each normal in G_1, such that*

(i) G_i/G_{i+1} *is abelian for all* $i \geq 1$*;*

(ii) $\lim_{i\to\infty}((\log[G_i : G_{i+1}])/[G : G_i]) = \infty$*;*

(iii) $\limsup_i (d(G_i/G_{i+1})/[G : G_i]) > 0$.

Then G_i is a non-trivial graph of groups for all sufficiently large i.

We first replace G by G_1, so that each G_i is normal in G. Since G_i/G_{i+1} is abelian for all $i \geq 1$, the following theorem of Lubotzky and Weiss [8] applies. This is, in fact, not exactly how they stated their result (which appears as Theorem 3.6 of [8]), but this formulation can readily be deduced from their argument.

Theorem. *Suppose that a finitely generated group G has Property (τ) with respect to*

a collection $\{G_i\}$ of finite index subgroups. Then there is a constant c with the following property. If $G_{i+1} \lhd G_i$ and G_i/G_{i+1} is abelian, then $|G_i/G_{i+1}| < c^{[G:G_i]}$.

Hence, by properties (i) and (ii), G does not have Property (τ) with respect to $\{G_i\}$. Since $\{G_i\}$ is a nested sequence, $(d(G_i) - 1)/[G : G_i]$ is a non-increasing function of i. So, the rank gradient of $(G, \{G_i\})$ is

$$\inf_i \left\{ \frac{d(G_i) - 1}{[G : G_i]} \right\} = \limsup_i \left\{ \frac{d(G_i) - 1}{[G : G_i]} \right\} \geq \limsup_i \left\{ \frac{d(G_i/G_{i+1})}{[G : G_i]} \right\},$$

which by (iii) is positive. So, the only possible conclusion of Theorem 1.1 is (1). Once we know that one G_i is a non-trivial graph of groups, the same is true of all its finite index subgroups. This proves Theorem 5.1. □

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Mathematical Institute, Oxford University,
24-29 St Giles', Oxford OX1 3LB, UK.